

Different forms of metric characterizations of classes of Banach spaces

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Abstract. For each sequence $\{X_m\}_{m=1}^\infty$ of finite-dimensional Banach spaces there exists a sequence $\{H_n\}_{n=1}^\infty$ of finite connected unweighted graphs with maximum degree 3 such that the following conditions on a Banach space Y are equivalent:

- Y admits uniformly isomorphic embeddings of $\{X_m\}_{m=1}^\infty$.
- Y admits uniformly bilipschitz embeddings of $\{H_n\}_{n=1}^\infty$.

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1 Introduction

In connection with problems of embeddability of metric spaces into Banach spaces it would be interesting to find metric characterizations of well-known classes of Banach spaces. By a *metric characterization* we mean a set of formulas with some variables, quantifiers and inequalities, where the inequalities are between algebraic expressions containing distances between those variables which are elements of spaces. We say that such set of formulas *characterizes* a class \mathcal{P} of Banach spaces if $X \in \mathcal{P}$ if and only if all of the formulas of the set hold for X . We consider a more narrow class of metric characterizations, namely characterizations based on the notion of test-spaces.

Definition 1.1. Let \mathcal{P} be a class of Banach spaces and let $T = \{T_\alpha\}_{\alpha \in A}$ be a set of metric spaces. We say that T is a set of *test-spaces* for \mathcal{P} if the following two conditions are equivalent

1. $X \notin \mathcal{P}$
2. The spaces $\{T_\alpha\}_{\alpha \in A}$ admit bilipschitz embeddings into X with uniformly bounded distortions.

Remark 1.2. Reading of the rest of this introduction requires more background than reading Sections 2–5 of the paper.

Remark 1.3. We write $X \notin \mathcal{P}$ rather than $X \in \mathcal{P}$ for terminological reasons: we would like to use terms “test-spaces for reflexivity, superreflexivity, etc.” rather than “test-spaces for **non**reflexivity, **nonsuper**reflexivity, etc.”

Remark 1.4. For each collection $\{T_\alpha\}_{\alpha \in A}$ of metric spaces the condition (2) of Definition 1.1 determines the corresponding class of Banach spaces. It seems that at least in some cases it would be interesting to start with a class of metric spaces and to try to understand what is the corresponding class of Banach spaces. Papers [CK09, Corollary 1.7] and [Ost11, Theorems 3.2 and 3.6] contain results of this type. We are not going to pursue this direction here.

Test-spaces are known for several important classes of Banach spaces: superreflexive [Bou86, Mat99, Bau07, JS09]; spaces having some type $t > p$, where $p \in [1, 2]$ [BMW86, Pis86, Bau07]; spaces with non-trivial cotype [MN08, Bau09+].

There are some test-space characterizations which are usually stated differently and considered as Banach-space-theoretical characterizations rather than metric characterizations. We mean the following characterizations. We refer to [DJT95] and [LT79] for the theory of type and cotype of Banach spaces.

Maurey and Pisier [MP76] introduced for each infinite-dimensional Banach space X parameters $p(X) = \sup\{p : X \text{ has type } p\}$ and $q(X) = \inf\{q : X \text{ has cotype } q\}$ and proved that the spaces $\{\ell_{p(X)}^n\}_{n=1}^\infty$ and $\{\ell_{q(X)}^n\}_{n=1}^\infty$ are finitely representable in X in the sense that $\forall \varepsilon > 0 \forall n \in \mathbb{N}$ there is a subspace $X_{n,\varepsilon}$ in X satisfying $d(X_{n,\varepsilon}, \ell_{p(X)}^n) \leq 1 + \varepsilon$, where d is the Banach-Mazur distance. (This result is based on the work of Krivine [Kri76], proofs of these results can be found in [MS86].)

On the other hand, Bretagnolle, Dacunha-Castelle, and Krivine [BDK66] proved that for $1 \leq p < q \leq 2$ the space L_q admits an isometric embedding into L_p .

Also, as is easy to check, the space L_p ($1 \leq p \leq 2$) has type p but does not have any type $t > p$ (see [DJT95, p. 216]).

Combining these results we get the following characterizations.

- (i) Let $p \in [1, 2)$. Let \mathcal{A}_p be the class of Banach spaces for which $p(X) > p$. Then $X \notin \mathcal{A}_p$ if and only if $\{\ell_p^n\}_{n=1}^\infty$ admit uniformly isomorphic embeddings into X .
- (ii) In a similar way spaces with $q(X) = \infty$ are characterized as spaces admitting uniformly isomorphic embeddings of $\{\ell_\infty^n\}_{n=1}^\infty$.

The characterizations just stated are not in terms of bilipschitz embeddings, but applying the theory of differentiability, see [HM82] and [BL00, Theorem 7.9 and Corollary 7.10] and local reflexivity (see [LR69] and [JRZ71]) it can be shown that the word “isomorphic” can be replaced by the word “bilipschitz” in (i)–(ii). (We say that a collection of embeddings is *uniformly bilipschitz* if the embeddings have uniformly bounded distortions.) We do not present this argument in detail here because it is almost identical to the argument which we present at the end of our proof of Theorem 2.1.

In this paper we are interested in the following problem.

Problem 1.5. *Let \mathcal{P} be a class of Banach spaces which can be characterized in terms of countably many test-spaces which are finite-dimensional normed spaces.*

- (a) *Is it possible to describe \mathcal{P} in terms of countably many test-spaces which are finite unweighted graphs with their graph distances?*
- (b) *Is it possible to require, in addition, that the graphs have uniformly bounded degrees of vertices?*

Special cases of this problem were posed by W.B. Johnson during the seminar “Nonlinear geometry of Banach spaces” (Workshop in Analysis and Probability at Texas A & M University, 2009).

Our main purpose is to give an affirmative answer to Problem 1.5.

2 Graphic test-spaces for classes having finite-dimensional test-spaces

Our first goal is to prove the following result:

Theorem 2.1. *If a class \mathcal{P} can be characterized using test-spaces $\{X_m\}_{m=1}^\infty$ which are finite dimensional Banach spaces, then \mathcal{P} can be characterized using test-spaces which are finite unweighted graphs with their graph distances.*

By a δ -net in a metric space Z we mean a collection U of elements of Z satisfying the conditions:

- $\forall z \in Z \exists u \in U d_Z(u, z) \leq \delta.$
- $\forall u, v \in U d_Z(u, v) \geq \delta.$

Lemma 2.2. *For each finite-dimensional Banach space X and each pair δ, r satisfying $0 < \delta < r < \infty$ there is a finite unweighted graph $G = (V(G), E(G))$ and a map $f : V(G) \rightarrow rB(X)$ ($rB(X)$ is a multiple of the unit ball) such that f is a bilipschitz embedding with distortion ≤ 3 and $f(V(G))$ is a δ -net in $rB(X)$ with distances $> \delta$ between images of different vertices.*

Remark 2.3. It is interesting to compare this lemma with the observation that an unweighted graph G which admits an isometric embedding into a strictly convex Banach space should be either a path or a complete graph. Most probably this observation is known. I enclose a proof of it in Section 5 because I have not found it in the literature.

Proof of Lemma 2.2. Since X is finite-dimensional, there is a finite δ -net $V = \{v_i\}_{i=1}^n$ in $rB(X)$ with $\|v_i - v_j\| > \delta$. We introduce a graph structure on V using the following rule: vertices v_i and v_j are adjacent if and only if $\|v_i - v_j\| \leq 3\delta$. Denote the obtained graph by $G = G(X, \delta, r)$.

Let $f : V \rightarrow X$ be the natural embedding (that is, embedding which maps each vertex onto itself). It is clear that $\text{Lip}(f) \leq 3\delta$ and that $f(V)$ is a δ -net in $rB(X)$. The only condition which is to be verified is $\text{Lip}(f^{-1}) \leq \delta^{-1}$. It suffices to show that

for each pair $u, v \in V$ satisfying $\|u - v\| = d\delta$, there is a uv -path having at most $\lfloor d \rfloor$ edges.

If $1 < d \leq 3$, the statement is obvious since u and v are adjacent (observe that d cannot be ≤ 1). If $d > 3$ we use the following lemma.

Lemma 2.4. *Let $u, v \in V$ be such that $\|u - v\| = d\delta > 3\delta$. Then there is a vertex $u_1 \in V$ satisfying $\|u - u_1\| \leq 3\delta$ and $\|u_1 - v\| \leq (d - 1)\delta$.*

Proof. Let w be the point satisfying $\|u - w\| = 2\delta$ and belonging to the line segment joining u and v . Then there is $u_1 \in V$ satisfying $\|u_1 - w\| \leq \delta$. By the triangle inequality we have $\|u - u_1\| \leq 3\delta$ and $\|v - u_1\| \leq \|v - w\| + \|u_1 - w\| \leq d\delta - 2\delta + \delta = (d - 1)\delta$. \square

We complete the proof of Lemma 2.2 using induction. We know that the claim holds when $\|u - v\| < 3\delta$.

Induction Hypothesis: The claim holds when $\|u - v\| < n\delta$.

Now assume that $n\delta \leq \|u - v\| < (n + 1)\delta$. We apply Lemma 2.4 and get u_1 satisfying $\|u - u_1\| \leq 3\delta$ and $\|u_1 - v\| \leq \|u - v\| - \delta < n\delta$. By the Induction Hypothesis there is a u_1v -path of length $\leq \lfloor \|u_1 - v\|/\delta \rfloor$. Also u and u_1 are adjacent in G . Adding this edge to the u_1v -path we get a uv -path of length $\leq \lfloor \|u_1 - v\|/\delta \rfloor + 1 \leq \lfloor \|u - v\|/\delta \rfloor$. \square

Proof of Theorem 2.1. Our purpose is to show that the countable collection

$$\{G(X_m, 1/n, n)\}_{m,n=1}^{\infty}$$

of graphs is the desired collection of test-spaces, where $\{X_m\}_{m=1}^{\infty}$ are finite-dimensional Banach test-spaces for \mathcal{P} .

It is clear that the spaces $\{G(X_m, \frac{1}{n}, n)\}_{m,n=1}^{\infty}$ admit uniformly bilipschitz embeddings into any Banach space admitting uniformly bilipschitz embeddings of $\{X_m\}_{m=1}^{\infty}$.

It remains to show that if a Banach space Y admits uniformly bilipschitz embeddings of $\{G(X_m, \frac{1}{n}, n)\}_{m,n=1}^{\infty}$, then there are uniformly isomorphic embeddings of the Banach spaces $\{X_m\}$ into Y . This can be proved in the following way (the author learned this argument from G. Schechtman, see [Bau09+, Proposition 4.2]).

Fix $m \in \mathbb{N}$. Let $f_n : G(X_m, \frac{1}{n}, n) \rightarrow Y$ be uniformly bilipschitz embeddings. We may assume that there is $0 < C < \infty$ such that $d_n(u, v) \leq \|f_n(u) - f_n(v)\| \leq Cd_n(u, v)$ for all vertices u, v of $G(X_m, \frac{1}{n}, n)$, where d_n is the graph distance of $G(X_m, \frac{1}{n}, n)$. We may and shall assume that the zero element of X_m is a vertex of $G(X_m, \frac{1}{n}, n)$, and that $f_n(0) = 0$, where the first 0 is the zero element of X_m and the second 0 is the zero element in Y .

We use the embeddings f_n to find a C -bilipschitz embedding of X_m into an ultrapower of X . For each y in X_m we introduce the following sequence $y_n \in V(G(X_m, \frac{1}{n}, n))$, $n \in \mathbb{N}$:

$$y_n = \begin{cases} 0 & \text{if } \|y\| > n \\ \text{best approximation of } y \text{ by elements of } V(G(X_m, \frac{1}{n}, n)) & \text{if } \|y\| \leq n. \end{cases} \quad (1)$$

In this definition, we pick one of the best approximations if there are several of them.

Let \mathcal{U} be a free ultrafilter on \mathbb{N} . It is easy to check that the mapping $F : X_m \rightarrow Y$ given by

$$F(y) = \left\{ \frac{1}{n} f_n(y_n) \right\}_{n=1}^{\infty}$$

is a $3C$ -bilipschitz embedding of X_m into $(Y)_{\mathcal{U}}$ and thus into the second dual $((Y)_{\mathcal{U}})^{**}$. By [BL00, Theorem 7.9 and Corollary 7.10] (these results go back to [HM82]), this implies an existence of C -isomorphic embedding of X_m into $((Y)_{\mathcal{U}})^{**}$. Using the local reflexivity ([LR69] and [JRZ71]) and standard properties of ultra-products (see [DK72] or [DJT95]) we get that $\{X_m\}_{m=1}^{\infty}$ are uniformly isomorphic to subspaces of Y . \square

3 Test-spaces with uniformly bounded degrees

It is easy to see that for a fixed finite-dimensional space X the graphs $\{G(X, \frac{1}{n}, n)\}$ have uniformly bounded degrees, but when we consider a family of graphs corresponding to spaces $\{X_m\}$ with growing dimensions, we get graphs with no uniform bound on degrees. Therefore Problem 1.5(b) is nontrivial in this case. However, the answer to it is positive:

Theorem 3.1. *Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of finite-dimensional normed spaces satisfying $\sup_m \dim X_m = \infty$. Then there exists a sequence $\{H_n\}_{n=1}^{\infty}$ of finite unweighted graphs with maximum degree 3 such that a Banach space Y admits uniformly bilipschitz embeddings of $\{H_n\}_{n=1}^{\infty}$ if and only if Y admits uniformly bilipschitz (or uniformly isomorphic) embeddings of $\{X_m\}_{m=1}^{\infty}$.*

Remark 3.2. It is easy to see that the result remains true if $\sup_m \dim X_m < \infty$, but this case seems to be of little interest.

The general scheme of the proof of Theorem 3.1 is the same as in [Ost11, Theorem 2.1]. The main step in the proof of Theorem 3.1 is the following lemma (its analogues for the cases considered in [Ost11] were much easier).

Lemma 3.3. *Let X be a finite-dimensional normed space with $\dim X \geq 3$ and let $G = G(X, \delta, r)$, $0 < \delta < r$, be a graph defined in the proof of Lemma 2.2. Let $M \in \mathbb{N}$ and let MG be the graph obtained from G if each edge is replaced by a path of length M . Then the graphs $\{MG\}_{M=1}^{\infty}$ admit uniformly bilipschitz embeddings*

into X . Furthermore, there is an upper bound on distortion which is an absolute constant.

First we prove Theorem 3.1 using Lemma 3.3. Since the sequence $\{\dim X_m\}_{m=1}^\infty$ is unbounded, a Banach space which admits uniformly isomorphic embeddings of $\{X_m\}_{m=1}^\infty$, admits uniformly isomorphic embeddings of the sequence $\{X_m \oplus_1 \mathbb{R}\}_{m=1}^\infty$. It is also clear that if we drop from the sequence $\{X_m\}$ all spaces with $\dim X_m < 3$, this would not change the class of Banach spaces admitting uniformly isomorphic embeddings of $\{X_m\}$. Therefore we may and shall assume that Lemma 3.3 is applicable to each member of the sequence $\{X_m\}$.

Our proof of Theorem 2.1 implies that it suffices to show that for each graph $G = G(X, \delta, r)$ there exist a graph H and bilipschitz embeddings $\psi : G \rightarrow H$ and $\varphi : H \rightarrow X \oplus_1 \mathbb{R}$ such that their distortions are bounded from above by absolute constants and the maximum degree of H is 3. It is easy to see that it is enough to consider the case $\delta = 1$.

Our construction can be described in the following way: First we expand G replacing each edge by a path of length M . We use the term *long paths* for these paths, the number M here is chosen to be much larger than the number of edges of G (actually we can replace the number of edges of G by a smaller number in this argument, but we do not see reasons to work on this modification now). Then, for each vertex v of G , we introduce a path p_v in the graph H (which we are constructing now) whose length is equal to the number of edges of G , we call each such path a *short path*. At the moment these paths do not interact. We continue our construction of H in the following way. We label vertices of short paths in a monotone way by long paths. “In a monotone way” means that the first vertex of **each** short path corresponds to the long path p_1 , the second vertex of **each** short path corresponds to the long path p_2 etc. We complete our construction of H introducing, for a long path p in MG corresponding to an edge uv in G , a path of the same length in H (we also call it *long*) which joins those vertices of the short paths p_u and p_v which have label p . There is no further interaction between short and long paths in H . It is obvious that the maximum degree of H is 3.

It remains to define embeddings ψ and φ and to estimate their distortions.

To define ψ we pick a long path p in MG (in an arbitrary way) and map each vertex u of G onto the vertex in H having label p in the short path p_u corresponding to u . We have $\text{Lip}(\psi) \leq 2e(G) + M$, where $e(G)$ is the number of edges of G . In fact, to estimate the Lipschitz constant it suffices to find an estimate from above for the distances in H between $\psi(u)$ and $\psi(v)$ where u and v are adjacent vertices of G . To see that $2e(G) + M$ provides the desired estimate we consider the following three-stage walk from $\psi(u)$ from $\psi(v)$:

- We walk from $\psi(u)$ along the short path p_u to the vertex labelled by the long path corresponding to the edge uv in G .
- Then we walk along the corresponding long path to its end in p_v .

- We conclude the walk with the piece of the short path p_v which we need to traverse in order to reach $\psi(v)$.

We claim that $\text{Lip}(\psi^{-1}) \leq M^{-1}$ (this gives an absolute upper bound for the distortion of ψ provided the quantity $e(G)$ does not exceed M , the assumption “ M is much larger than $e(G)$ ” made above is needed only if we would like to make the distortion close to 1). In fact, let $\psi(u)$ and $\psi(v)$ be two vertices of $\psi(V(G))$. We need to estimate $d_G(u, v)$ from below in terms of $d_H(\psi(u), \psi(v))$. Let $P = \psi(u), w_1, \dots, w_n = \psi(v)$ be one of the shortest $\psi(u)\psi(v)$ -paths in H . Let $u, u_1, \dots, u_k = v$ be those vertices of G for which the path P visits the corresponding short paths $p_u, p_{u_1}, \dots, p_{u_k} = p_v$. We list u_1, \dots, u_k in the order of visits. It is clear that in such a case $u, u_1, \dots, u_k = v$ is a uv -walk in G . Therefore $d_G(u, v) \leq k$. On the other hand, in H , to move from one short path to another, one has to traverse at least M edges, therefore $d_H(\psi(u), \psi(v)) \geq kM$. This implies $\text{Lip}(\psi^{-1}) \leq M^{-1}$.

Now we describe $\varphi : H \rightarrow X \oplus_1 \mathbb{R}$. First we recall that by Lemma 3.3 there is a bilipschitz embedding of MG into X , we denote this embedding by φ_0 . We may and shall assume that $\text{Lip}(\varphi_0) \leq 1$ and $\text{Lip}(\varphi_0^{-1})$ is bounded from above by an absolute constant. Next, we number vertices along short paths using numbers from 1 to $e(G)$ in such a way that vertices numbered 1 correspond to the same long path in the correspondence described above.

At this point we are ready to describe the action of the map φ on vertices of short paths. For vertex w of H having number i on the short path p_u the image in $X \oplus_1 \mathbb{R}$ is $\varphi_0(u) \oplus i$ (here we use the same notation u both for a vertex of G and the corresponding vertex in MG).

To map vertices of long paths of H into $X \oplus_1 \mathbb{R}$ we observe that the numbering of vertices of short paths leads to a one-to-one correspondence between long paths and numbers $\{1, \dots, e(G)\}$. We define the map φ on a long path corresponding to i by $\varphi(w) = \varphi_0(w') \oplus i$, where w' is the uniquely determined vertex in a long path of MG corresponding to a vertex w in a long path of H .

The fact that $\text{Lip}(\varphi) \leq 1$ follows immediately from the easily verified claim that the distance between φ -images of adjacent vertices of H is at most 1 (here we use $\text{Lip}(\varphi_0) \leq 1$).

We turn to an estimate of $\text{Lip}(\varphi^{-1})$. In this part of the proof we assume that $M > 2e(G)$. Let w and z be two vertices of H . As we have already mentioned our construction implies that there are uniquely determined corresponding vertices w' and z' in MG .

Obviously there are two possibilities:

(1) $d_{MG}(w', z') \geq \frac{1}{2}d_H(w, z)$. In this case we observe that the definitions of φ and of the norm on $X \oplus_1 \mathbb{R}$ imply that

$$\|\varphi(w) - \varphi(z)\| \geq \|\varphi_0(w') - \varphi_0(z')\| \geq d_{MG}(w', z') / \text{Lip}(\varphi_0^{-1}) \geq \frac{1}{2}d_H(w, z) / \text{Lip}(\varphi_0^{-1}).$$

(2) $d_{MG}(w', z') < \frac{1}{2}d_H(w, z)$. This inequality implies that there is a path joining w and z for which the naturally defined *short-paths-portion* is longer than the *long-paths-portion*. The inequality $M > 2e(G)$ implies that the short-paths-portion of this path consists of one path of length $> \frac{1}{2}d_H(w, z)$. This implies that the difference between the second coordinates of w and z in the decomposition $X \oplus_1 \mathbb{R}$ is $> \frac{1}{2}d_H(w, z)$. Thus $\|\varphi(w) - \varphi(z)\| > \frac{1}{2}d_H(w, z)$.

Since $\text{Lip}(\varphi_0^{-1}) \geq 1$ (this follows from the assumption $\text{Lip}(\varphi_0) \leq 1$), we get $\text{Lip}(\varphi^{-1}) \leq 2\text{Lip}(\varphi_0^{-1})$ in each of the cases (1) and (2). \square

4 Proof of Lemma 3.3

Proof. It is clear that it suffices to consider the case $\delta = 1$. It is convenient to handle all $M \in \mathbb{N}$ simultaneously by considering the following thickening of the graph $G = G(X, 1, r)$ (see [Gro93, Section 1.B] for the general notion of thickening). For each edge uv in G we join u and v with a set isometric to $[0, 1]$, we denote this set $t(uv)$. The thickening TG is the union of all sets $t(uv)$ (such sets can intersect at their ends only) with the distance defined as the length of the shortest curve joining the points.

In order to prove Lemma 3.3 it suffices to show that there is a bilipschitz embedding of TG into X with distortion bounded by an absolute constant. This follows immediately from the observation that the graph MG with the scaled distance $\frac{1}{M}d_{MG}(u, v)$ is isometric to a subset of TG .

Recall that vertices of G are elements of X . The restriction of our embedding of TG into X to $V(G)$ will be the identical map. So we need to define the embedding for edges only. We start by looking at the following straightforward approach: map $t(uv)$ onto the line segment $[u, v]$ in such a way that the point in $t(uv)$ which is at distance $\alpha \in (0, 1)$ from u is mapped onto the point $\alpha u + (1 - \alpha)v \in [u, v]$ (where $[u, v]$ denotes the line segment joining u and v in X). It is clear that in general this straightforward approach does not have to work: it can happen that $[u, v]$ intersects the line segment $[u_1, v_1]$ corresponding to some other edge. In such a case the straightforward embedding is not bilipschitz. Recall also that we need to bound the distortion of the bilipschitz embedding of TG into X by an absolute constant.

It turns out that the following perturbation of the straightforward construction works. Recall that $\delta = 1$. Let $\mu = \frac{1}{4}$ and let z be the midpoint of $[u, v]$. Let $B(z, \mu)$ denote the ball of radius μ centered at z . Our purpose is to show that $B(z, \mu)$ contains a point w , such that mapping the edge $t(uv)$ onto the union of line segments $[u, w]$ and $[w, v]$, we get a bilipschitz embedding whose distortion is bounded by an absolute constant. Here we mean the map which maps the point of $t(uv)$ lying at distance α from u onto the point in the curve obtained by concatenation of the line segments $[u, w]$ and $[w, v]$ which is at along-the-curve distance $\alpha(\|w - u\| + \|v - w\|)$ from u .

The map defined in this way is 4-Lipschitz because it is clear that $\|w - u\| + \|v - w\| \leq \|z - u\| + \frac{1}{4} + \|v - z\| + \frac{1}{4} \leq 3.5 < 4$.

To get a suitable estimate for the Lipschitz constant of the inverse map we need to find w in such a way that the line segments $[u, w]$ and $[w, v]$ do not pass “too close” to line segments corresponding to other edges.

We make the notion of “not-too-close” more precise as follows. We pick three numbers α , β , and γ in the interval $(0, \frac{1}{4})$ satisfying $\alpha, \gamma < \beta < \mu$. Some more restrictions will be specified later, we also will show that numbers satisfying the restrictions exist and can be chosen independently of X and r .

The “not-too-close” condition will be understood in the following way. We find curves corresponding to different edges one by one. When we turn to the construction of the curve corresponding to the edge $t(uv)$, we do this in such a way that the following conditions are satisfied:

- (α) The intersection of the curve corresponding to $t(uv)$ with $B(u, \beta)$ is a line segment, and the distance from the intersection of this line segment with the sphere $S(u, \beta)$ (of radius β centered at u) to intersections with $S(u, \beta)$ with the line segments corresponding to previously embedded edges is at least α ; and the same condition holds for v .
- (β) The curve corresponding to $t(uv)$ does not intersect $B(\tilde{u}, \beta)$ for all vertices \tilde{u} other than u and v .
- (γ) The γ -neighborhoods of those parts of curves corresponding to edges which are not in the β -balls centered at vertices do not intersect curves corresponding to other edges.

The first part of our proof consists of showing that if the numbers α , β , and γ are sufficiently small (but this “smallness” is independent on X and its dimension provided $\dim X \geq 3$) and satisfy certain relations, then the set of points $x \in B(z, \mu)$ for which at least one of the line segments $[u, x]$ and $[x, v]$ does not meet the conditions above does not exhaust $B(z, \mu)$. More precisely, we show that the volume of the set of not-suitable points is smaller than the volume of $B(z, \mu)$.

The second part of the proof consists in showing that conditions (α)–(γ) imply an absolute-constant upper estimate of the distortion of the constructed map of TG into X .

Let $[u, \tilde{w}]$ be one of the line segments in the image of an already embedded edge $t(u\tilde{v})$ (recall that we map each edge onto a union of two line segments). Our first goal is to estimate from above the volume of the set of those points $x \in B(z, \mu)$ for which the line segments $[u, x]$ and $[u, \tilde{w}]$ violate the condition (α). Observe that

$$\frac{1}{2}\|v - u\| - \mu \leq \|x - u\| \leq \frac{1}{2}\|v - u\| + \mu < 2. \quad (2)$$

If condition (α) is violated, then $\|x - y\| \leq \alpha \cdot \frac{\|x - u\|}{\beta}$ for some y in the ray \overrightarrow{uw} (this is our notation for the ray which starts at u and passes through \tilde{w}) satisfying $\|y - u\| = \|x - u\|$. Therefore all vectors $x \in B(z, \mu)$ which are “too close” to the ray \overrightarrow{uw} , are contained in the $\frac{2\alpha}{\beta}$ -neighborhood T_1 of a line segment of length at most 2μ (contained in the ray \overrightarrow{uw}).

Convention. Everywhere in this proof by a *volume* of a subset of X we mean its Haar measure normalized in such a way that the volume of the unit ball $B(0, 1)$ is equal to 1.

To estimate the volume of the neighborhood T_1 we observe that a line segment of length 2μ has a $\frac{2}{\beta}\alpha$ -net of cardinality $\lceil \frac{\mu\beta}{2\alpha} \rceil$. As we shall see later, we have enough freedom in choosing α , to assume that $\frac{\mu\beta}{2\alpha}$ is an integer.

The triangle inequality implies that the union of balls with radii $\frac{4\alpha}{\beta}$ centered at all elements of the net covers T_1 . Hence

$$\text{vol}(T_1) \leq \frac{\mu\beta}{2\alpha} \cdot \left(\frac{4\alpha}{\beta} \right)^n = 2\mu \cdot \left(\frac{4\alpha}{\beta} \right)^{n-1},$$

where n is the dimension of X .

Now we estimate from above the number of sets T_1 of the described type which should be avoided when we try to find a suitable location for $t(uv)$. It is clear that the number of such sets is estimated from above by the degree of u in G . The estimate of the degree is standard (see e.g. [MS86, Lemma 2.6]): We need to estimate from above the number N of 1-separated points in a ball of radius 3 (see the definition of $G(X, \delta, r)$). Interiors of balls of radii $\frac{1}{2}$ centered at 1-separated points do not intersect and are inside the ball of radius $3 + \frac{1}{2} = \frac{7}{2}$. Hence $N \left(\frac{1}{2}\right)^n \leq \left(\frac{7}{2}\right)^n$ and $N \leq 7^n$.

Thus the volume of the set which we have to exclude from $B(z, \mu)$ in order to satisfy the condition (α) for both u and v is

$$\leq 2 \cdot 7^n \cdot (2\mu) \cdot \left(\frac{4\alpha}{\beta} \right)^{n-1}.$$

We would like this quantity to be less than $\frac{1}{4}\text{vol}(B(z, \mu))$. This is achieved if

$$\frac{1}{4}\mu^n \geq 2 \cdot 7^n \cdot (2\mu) \cdot \left(\frac{4\alpha}{\beta} \right)^{n-1}.$$

Since $n \geq 3$ and $\mu = \frac{1}{4}$, it is easy to verify that any pair α, β satisfying

$$\alpha \leq \frac{1}{1232} \beta, \tag{3}$$

satisfies the condition above.

Now we estimate from above the volume of the set of points $x \in B(z, \mu)$ for which the curve obtained by concatenation of the line segments $[u, x]$ and $[x, v]$ does not satisfy the condition (β) . Recall that any vertex y of $G = G(X, 1, r)$ which is different from u , is not contained in $B(u, 1)$. Therefore, if $\|y - \tilde{x}\| \leq \beta$ for some $\tilde{x} \in [u, x]$, then $\|\tilde{x} - u\| \geq (1 - \beta)$. Since $\|x - u\| \leq 2$ (see (2)), we get that the distance between x and some point on the ray $\overrightarrow{u\tilde{y}}$ is $\leq \frac{2}{1-\beta} \cdot \beta$. Therefore the set of points in $B(z, \mu)$ which violates the condition (β) for given $y \in V(G)$ is covered by $\frac{2}{1-\beta} \cdot \beta$ -neighborhood T_2 of a line segment of length $2\mu + 2 \left(\frac{2\beta}{1-\beta} \right)$. Our estimate of $\text{vol}(T_2)$ is similar to the estimate of $\text{vol}(T_1)$. Let us sketch it briefly. To simplify the estimate we assume that $\beta < \mu/5$. In such a case $2\mu + 2 \left(\frac{2\beta}{1-\beta} \right) < 3\mu$ and $\left(\frac{2\beta}{1-\beta} \right) < 3\beta$. So it suffices to estimate the volume of 3β -neighborhood of the line segment of length 3μ . In the same way as before we get the estimate $\leq \frac{3\mu}{6\beta} \cdot (6\beta)^n$.

We need to estimate the number of vertices y for which such sets have to be excluded. Here, for simplicity we can use the same estimate $\leq 7^n$ because it is clear that a vertex whose distance from u in X is > 3 cannot “stay on the way” of the line segment $[u, x]$, $x \in B(z, \mu)$.

We get the following upper estimate of the volume of the part of $B(z, \mu)$ which should be excluded in order to eliminate all points x for which the curve obtained by concatenation of $[u, x]$ and $[x, v]$ violates the condition (β) . The volume does not exceed $2 \cdot 7^n \cdot (3\mu) \cdot (6\beta)^{n-1}$. Again we would like this quantity to be less than $\frac{1}{4} \text{vol}(B(z, \mu)) = \frac{1}{4} \mu^n$. Recalling that $n \geq 3$ we get that the condition is satisfied for

$$\beta \leq \mu/546. \quad (4)$$

This is our second requirement on the triple (α, β, γ) .

Now we turn to estimates the volume of the sets of those $x \in B(z, \mu)$ for which the concatenation of $[u, x]$ and $[x, v]$ does not satisfy condition (γ) . This happens only if either $[u, x]$ or $[x, v]$ intersects a γ -neighborhood of an already embedded into X edge $f(t(\tilde{u}\tilde{v}))$ (f denotes the embedding). We do the estimates for the case when $[u, x]$ intersects a γ -neighborhood of an already embedded into X edge $f(t(\tilde{u}\tilde{v}))$; at the end we multiply the obtained estimate for the volume of non-suitable points by 2. First we estimate the number of such edges $\tilde{u}\tilde{v}$. In the described situation both \tilde{u} and \tilde{v} should be in a ball of radius 5 centered at u (recall that $\gamma \leq \frac{1}{4}$). In the same way as we estimated the number of vertices in a 3-ball, we get an estimate $\leq 11^n$ for the number of vertices \tilde{u} for which $f(t(\tilde{u}\tilde{v}))$ can occur as an obstacle. Hence the number of edges which could interfere with the line segment $[u, x]$ is $\leq 121^n$.

Now we estimate from above the volume of those $x \in B(z, \mu)$ which have to be excluded because of one already embedded edge $f(t(\tilde{u}\tilde{v}))$. First recall that the length of each such edge is ≤ 4 and that we do not need to care about its portion which is inside $B(u, \beta)$.

For each point $y \in f(t(\widetilde{uv}))$ satisfying $\|y - u\| \geq \beta$ we consider the ray \overrightarrow{uy} . If y and $[u, x]$ violate the condition (γ) , then there is $\tilde{x} \in [u, x]$ such that $\|y - \tilde{x}\| \leq \gamma$. Then $\|\tilde{x} - u\| \geq \beta - \gamma$. Since $\|x - u\| \leq 2$, we get that the distance between x and some point on the ray \overrightarrow{uy} is $\leq \frac{2}{\beta - \gamma} \cdot \gamma$. Therefore the set of points in $B(z, \mu)$ for which $[u, x]$ comes “too close” to the point $y \in f(t(\widetilde{uv}))$ is covered by $\frac{2\gamma}{\beta - \gamma}$ -neighborhood $T_3(y)$ of a line segment of length $\leq 2\mu + \frac{4\gamma}{\beta - \gamma}$ on the ray \overrightarrow{uy} .

For simplicity of the remaining argument we assume that $\gamma \leq \frac{1}{20}\beta$. In such a case $\frac{4\gamma}{\beta - \gamma} \leq \mu$ and $\frac{2\gamma}{\beta - \gamma} \leq \frac{3\gamma}{\beta}$. Thus $T_3(y)$ is covered by a $\frac{3\gamma}{\beta}$ -neighborhood of a line segment $\ell(y)$ of length 3μ . Furthermore, it is easy to see that we may assume that the center of $\ell(y)$ is at distance $\frac{1}{2}\|v - u\|$ from u . We need to estimate from above the volume of

$$T_3 := \bigcup_{y \in f(t(\widetilde{uv}))} T_3(y).$$

We need the following observation: Let h_1 and h_2 be two vectors in X satisfying $\|h_1\| \geq \|h_2\| > 0$. By the triangle inequality we have

$$\left\| h_1 - \frac{\|h_1\|}{\|h_2\|} h_2 \right\| \leq 2\|h_1 - h_2\|. \quad (5)$$

Let y_1 and y_2 be two points on the curve $f(t(\widetilde{uv}))$ with $\|y_1 - u\| \geq \|y_2 - u\| \geq \beta$. Using (5) for $h_1 = y_1 - u$ and $h_2 = y_2 - u$ and homothety we get that the distance between a point of $\ell(y_1)$ and the point of $\ell(y_2)$ with the same distance to u is

$$\leq \frac{2}{\beta} \cdot 2\|y_1 - y_2\| = \frac{4}{\beta} \|y_1 - y_2\|. \quad (6)$$

Since the length of $f(t(\widetilde{uv}))$ is ≤ 4 , there is a γ -net of cardinality $\leq \frac{4}{2\gamma}$ in $f(t(\widetilde{uv}))$. Also there is a $\frac{\gamma}{\beta}$ -net of cardinality $\leq \frac{3\mu\beta}{2\gamma}$ in any line segment of length 3μ . Combining these nets and using inequality (6) we get a set \mathcal{M} satisfying the following conditions: (a) $|\mathcal{M}| \leq \frac{3\mu\beta}{\gamma^2}$; (b) For each $q \in \bigcup_{y \in f(t(\widetilde{uv}))} \ell(y)$ there is $q_0 \in \mathcal{M}$ with $\|q - q_0\| \leq \frac{5\gamma}{\beta}$.

Therefore balls of radii $\frac{8\gamma}{\beta}$ centered at elements of \mathcal{M} cover T_3 . We get the estimate

$$\text{vol}(T_3) \leq \frac{3\mu\beta}{\gamma^2} \cdot \left(\frac{8\gamma}{\beta} \right)^n.$$

As we have already mentioned, we should consider $\leq 121^n$ already embedded edges, also we need to consider the line segments $[x, v]$ as well. Thus, the volume of all points x in $B(z, \mu)$ which fail to satisfy (γ) is estimated from above by

$$2 \cdot 121^n \cdot \frac{3\beta\mu}{\gamma^2} \cdot \left(\frac{8\gamma}{\beta} \right)^n.$$

As in the previous estimates, we would like this quantity to be $\leq \frac{1}{4}\text{vol}(B(z, \mu))$, that is, we need the inequality

$$2 \cdot 121^n \cdot \frac{3\beta\mu}{\gamma^2} \cdot \left(\frac{8\gamma}{\beta}\right)^n \leq \frac{1}{4} \mu^n$$

to hold. This inequality can be rewritten as

$$\gamma^{n-2} \leq C \mu^{n-1} \beta^{n-1} \left(\frac{1}{8 \cdot 121}\right)^{n-2},$$

where C is an absolute constant.

Since $n \geq 3$ we have $n-2 \geq 1$ and $n-1 \leq 2(n-2)$. Because of this and because $\beta, \mu \in (0, 1)$, we have $(\beta\mu)^{n-1} \geq (\beta\mu)^{2(n-2)}$. Therefore it suffices to satisfy

$$\gamma^{n-2} \leq C \left(\frac{(\beta\mu)^2}{8 \cdot 121}\right)^{n-2}.$$

Using again the inequality $n \geq 3$, we see that it suffices to pick

$$\gamma \leq C \frac{(\beta\mu)^2}{8 \cdot 121}. \quad (7)$$

Now it is clear that we can choose α , β , and γ satisfying the conditions (3), (4), and (7). We start with choosing β satisfying (4), then we choose α satisfying (3) and γ satisfying (7). It is clear that we may assume $\gamma \leq \alpha$ and that we had right to make the assumptions on relations between α , β , and γ which we made in our proof.

Thus, at each step it is possible to find $w \in B(z, \mu)$ such that the curve obtained by concatenation of $[u, w]$ and $[w, v]$ satisfies the assumptions (α) – (γ) .

To complete the proof of Lemma 3.3 it suffices to estimate the Lipschitz constant of the inverse map by an absolute constant. So we need to consider two points u, v in the image of TG and to estimate from above the ratio

$$\frac{d_{TG}(u, v)}{\|f(u) - f(v)\|}. \quad (8)$$

The estimate in the case when both points are vertices is given in Lemma 2.2. Next we consider the case when one of the points, say u , is a vertex. There are two subcases: (1) The second point is on the edge incident to u ; (2) The second point is on the edge which is not incident to u .

Subcase (1): If $\|f(u) - f(v)\| \leq \beta$, then the corresponding portion of the edge is a line segment. Since edges of length 1 are represented by curves whose length is > 1 , this implies $d_{TG}(u, v) \leq \|f(u) - f(v)\|$. If $\|f(u) - f(v)\| \geq \beta$, then, since the distance in TG is ≤ 1 , the ratio (8) is $\leq \frac{1}{\beta}$.

Subcase (2): The second point v is not on an edge incident with u . Let $D = \|f(u) - f(v)\|$. Then the distance in X between u and one of the ends of the edge to which v belongs is $\leq D + 2$ (recall that vertices of G are identified with elements of X , so $u = f(u)$). By the proof of Lemma 2.2, there is a path from that end to u of length $\leq D + 2$. Hence $d_{TG}(u, v) \leq D + 3$. Since by condition (β) we have $D \geq \beta$, the ratio (8) in this case does not exceed

$$\frac{D + 3}{D} \leq \frac{\beta + 3}{\beta} = 1 + \frac{3}{\beta}.$$

Now we consider the situation when u and v are on different edges. Subcases: (1) The edges are adjacent; (2) The edges are not adjacent.

Subcase (1): Subsubcase (a) One of the points is outside the β -ball centered at the common end. Then, by condition (γ) , we have $\|f(u) - f(v)\| \geq \gamma$. Since in TG the distance between two points belonging to adjacent edges is ≤ 2 , we get that the ratio (8) is $\leq \frac{2}{\gamma}$.

Subsubcase (b) Both points are inside the β -ball centered at some vertex $q = f(q)$. Let $\|f(u) - f(q)\| \leq \|f(v) - f(q)\| = \omega\beta$ for some $\omega \in (0, 1]$. Using condition (α) and a simple geometric argument we get that the distance between $f(v)$ and the ray $\overrightarrow{f(q)f(u)}$ is $\geq \frac{\alpha\omega}{2}$. On the other hand, $d_{TG}(u, v) \leq 2\omega\beta$. Therefore the ratio (8) in this case is $\leq \frac{2\omega\beta}{\frac{\alpha\omega}{2}} = \frac{4\beta}{\alpha}$.

Subcase (2) The points u, v are on non-adjacent edges. Then $\|f(u) - f(v)\| \geq \gamma$. Let $D = \|f(u) - f(v)\|$. Then the distance between two of the ends of the edges in X is $\leq D + 4$. Hence in TG it is also $\leq D + 4$. Hence the total distance between the points in the graph is $\leq D + 6$, and the ratio (8) is

$$\leq \frac{D + 6}{D} \leq \frac{\gamma + 6}{\gamma} \leq 1 + \frac{6}{\gamma}.$$

Taking into account the fact that we may assume that $\gamma \leq \alpha \leq \beta < 1$, we get that the Lipschitz constant of the inverse map is $\leq 1 + \frac{6}{\gamma}$. \square

5 Unweighted graphs admitting isometric embeddings into strictly convex Banach spaces

Observation 5.1. *If a finite simple connected graph G endowed with its graph distance admits an isometric embedding into a strictly convex Banach space X , then G is isomorphic to either a complete graph or a path.*

Proof. Assume that G is a finite simple connected graph, which is not a path, but is such that $(V(G), d_G)$ is isometric to a subset of a strictly convex space X (we use the standard definition of strict convexity, see [BL00, p. 409]). Denote the

isometric embedding by f . Our goal is to show that these conditions imply that G is a complete graph.

The fact that G is not a path immediately implies that G is either a cycle or has a vertex of degree 3. In the case when G is a cycle we observe that the cycle C_3 is simultaneously a complete graph K_3 , and we are done in this case. As for longer cycles we prove that they do not admit isometric embeddings into X in the following way. Since vertices v_{k-1}, v_k, v_{k+1} in a cycle satisfy $d_G(v_{k-1}, v_{k+1}) = d_G(v_{k-1}, v_k) + d_G(v_k, v_{k+1})$, by strict convexity we get that $f(v_{k-1}), f(v_k)$, and $f(v_{k+1})$ should be on the same line, with $f(v_k)$ being a midpoint of the line segment $[f(v_{k-1}), f(v_{k+1})]$. Since this observation is applicable also to v_{n-1}, v_n, v_1 and v_n, v_1, v_2 , we get a contradiction.

Now let $v \in V(G)$ be a vertex of degree ≥ 3 , and let u_1, u_2, u_3 be its neighbors. We show that u_i are pairwise adjacent. If two pairs of them (say u_1, u_2 and u_2, u_3) are not adjacent, we get a contradiction because $f(v)$ should be simultaneously a midpoint of the line segment joining $f(u_1)$ and $f(u_2)$ and a midpoint of the line segment joining $f(u_2)$ and $f(u_3)$.

If only one edge, say $u_1 u_3$, is missing then both $f(u_2)$ and $f(v)$ should be midpoints of the line segment joining $f(u_1)$ and $f(u_3)$.

Therefore v and all of its neighbors should form a complete subgraph in G . Since the same should hold for each of the neighbors of v , we get that G should be a complete graph. \square

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